On Non-homogeneous Ternary Bi-quadratic Equation $11(x+y)^2 = 4(xy+11z^4)$

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***Abstract* — This paper focusses on finding non-zero distinct integer solutions to the non-homogeneous ternary bi-quadratic equation $11(x+y)^2 = 4(xy+11z^4)$.**

Keywords — Ternary bi-quadratic; Non-homogeneous bi-quadratic; Integral solutions

I. INTRODUCTION

The Diophantine equations are rich in variety and offer an unlimited field for research [1-3]. In particular refer [4-15] for a few problems on Biquadratic equation with 3 unknowns. This paper concerns with yet another interesting Biquadratic Diophantine equation with three variables given by $11(x+y)^2 = 4(xy+11z^4)$ for determining its infinitely many non-zero distinct integral solutions

II. METHOD OF ANALYSIS

The non-homogeneous ternary bi-quadratic equation to be solved is

$$11(x+y)^2 = 4(xy+11z^4) \quad (1)$$

Introduction of the linear transformations

$$x = u+v, y = u-v, u \neq v \neq 0 \quad (2)$$

in (1) leads to

$$v^2 + 10u^2 = 11z^4 \quad (3)$$

The process of obtaining non-zero distinct integer solutions to (1) is illustrated below:

Illustration 1:

Write (3) in the form of ratio as

$$\frac{v+z^2}{z^2+u} = \frac{10(z^2-u)}{v-z^2} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (4)$$

$$x = 22\alpha\beta, y = 20\beta^2 - 2\alpha^2 - 18\alpha\beta \quad (5)$$

and

$$z^2 = \alpha^2 + 10\beta^2 \quad (6)$$

Now, (6) is satisfied by

$$\beta = 2rs, \alpha = 10r^2 - s^2 \quad (7)$$

and

$$z = 10r^2 + s^2 \quad (8)$$

Using (7) in (4),we have

$$x = 44rs, y = -200r^4 - 2s^4 + 120r^2s^2 - 36rs(10r^2 - s^2) \quad (9)$$

Thus, (8) and (9) represent the integer solutions to (1).

Note 1:

Observe that (3) may also be represented in the form of ratios as below:

(i) $\frac{v+z^2}{10(z^2+u)} = \frac{(z^2-u)}{v-z^2} = \frac{\alpha}{\beta}, \beta \neq 0$

(ii) $\frac{v+z^2}{2(z^2+u)} = \frac{5(z^2-u)}{v-z^2} = \frac{\alpha}{\beta}, \beta \neq 0$

(iii) $\frac{v+z^2}{5(z^2+u)} = \frac{2(z^2-u)}{v-z^2} = \frac{\alpha}{\beta}, \beta \neq 0$

Repeating the above process,three more integer solutions to (1) are found.

Illustration 2:

Taking

$$z^2 = X + 10T, u = X + 11T \quad (10)$$

in (3),it is written as

$$X^2 = 110T^2 + v^2 \quad (11)$$

Express (11) as the system of double equations as below in Table 1:

Table 1:System of double equations

System	I	II	III	IV
$X+v$	110T	55T	22T	11T
$X-v$	T	2T	5T	10T

Solving each of the above system of equations, the values of X, v, T are obtained. Then, from (10) and (2),the solutions to (1) are obtained .For brevity, the corresponding solutions are exhibited below:

Solutions from system I :

$$x=131*242*a^2, y=131*24*a^2, z=131a$$

Solutions from system II :

$$x=132*77*a^2, y=77*26*a^2, z=77a$$

Solutions from system III :

$$x=66*47*a^2, y=47*32*a^2, z=47a$$

$$x=41*44*a^2, y=41*42*a^2, z=41a$$

Note 2:

Apart from (10),one may also take

$$z^2=X-10T, u=X-11T$$

giving four more sets of solutions to (1).

Illustration 3:

Assume

$$z=a^2+10b^2 \quad (12)$$

Write 11 as

$$11=(1+i\sqrt{10})(1-i\sqrt{10}) \quad (13)$$

Substituting (12) & (13) in (3) and applying the method of factorization consider

$$v+i\sqrt{10}u=(1+i\sqrt{10})(a+i\sqrt{10}b)^4 \quad (14)$$

Equating the real and imaginary parts in (14) and using (2),we have

$$\left.\begin{aligned} x&=2(a^4-60a^2b^2+100b^4)-6(4a^3b-40ab^3),\\ y&=11(4a^3b-40ab^3)\end{aligned}\right\} \quad (15)$$

Thus,(12) and (15) give the integral solutions to (1).

Illustration 4:

Write (3) as

$$v^2+10u^2=11z^4*1 \quad (16)$$

Take 1 as

$$1=\frac{(3+i2\sqrt{10})(3-i2\sqrt{10})}{49} \quad (17)$$

Substituting (12),(13),(17) in(16) and employing the method of factorization, consider

$$v+i\sqrt{10}u=\frac{(-17+i5\sqrt{10})(a+i\sqrt{10}b)^4}{7} \quad (18)$$

Equate the real and imaginary parts in (18).Replace a by 7A,b by 7B in the resulting values of u and v. In view of (2), one has the integer values of x ,y and z satisfying (1) to be given by

$$\begin{aligned} x&=343[-12(A^4-60A^2B^2+100B^4)-67(4A^3B-40AB^3)],\\ y&=343[22(A^4-60A^2B^2+100B^4)+33(4A^3B-40AB^3)],\\ z&=49(A^2+10B^2)\end{aligned}$$

Note 3:

The integer 1 on the R.H.S. of (16) is also represented as below:

$$1=\frac{(3+i4\sqrt{10})(3-i4\sqrt{10})}{49},$$

$$1=\frac{(1+i6\sqrt{10})(1-i6\sqrt{10})}{361}$$

Repeating the above process,two more integer solutions to (1) are found.

Conclusion

An attempt has been made to obtain non-zero distinct integer solutions to the non-homogeneous bi-quadratic diophantine equation with three unknowns given by $11(x+y)^2=4(xy+11z^4)$.

One may search for other sets of integer solutions to the considered equation as well as other choices of the fourth degree diophantine equations with multi-variables

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